Howson property for monogenic inverse semigroups and the finitely generated intersection problem

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Joint work with Craig Miller and Nik Ruškuc

4th February 2026

York Semigroups

Howson property and inverse semigroups

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Equivalently, $(S, \cdot)$ is an inverse semigroup if it is regular, i.e. for every $s \in S$ there is $t \in S$ such that $sts = s$, and its idempotents commute.

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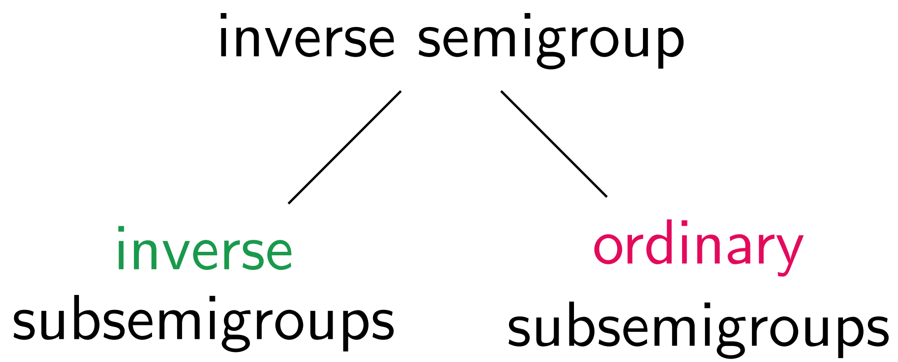

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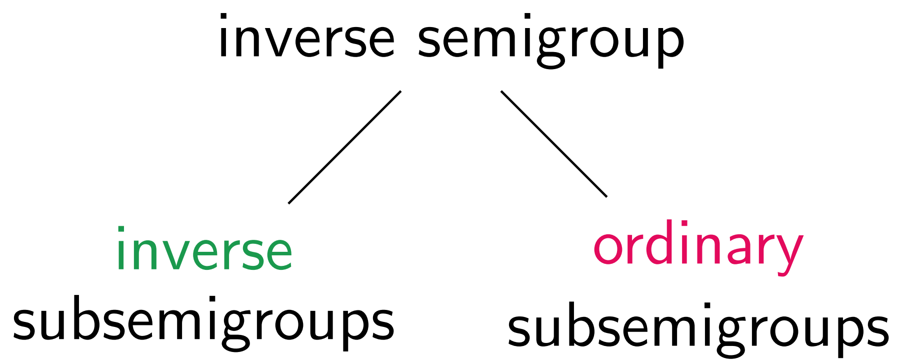

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inverse semigroup

inverse subsemigroups | ordinary subsemigroups

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- Semigroup Howson property: if the intersection of any two finitely generated ordinary (i.e. not necessarily inverse) subsemigroups is finitely generated.

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Theorem (Miller, Ruškuc, C., 25+)

A monogenic inverse semigroup has the semigroup Howson property if and only if it is not free.

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1	x	x^2	x^3	
x^{-1}	$x^{-1}x$	$x^{-1}x^2$	$x^{-1}x^3$	$\cdots$
x^{-2}	$x^{-2}x$	$x^{-2}x^2$	$x^{-2}x^3$	
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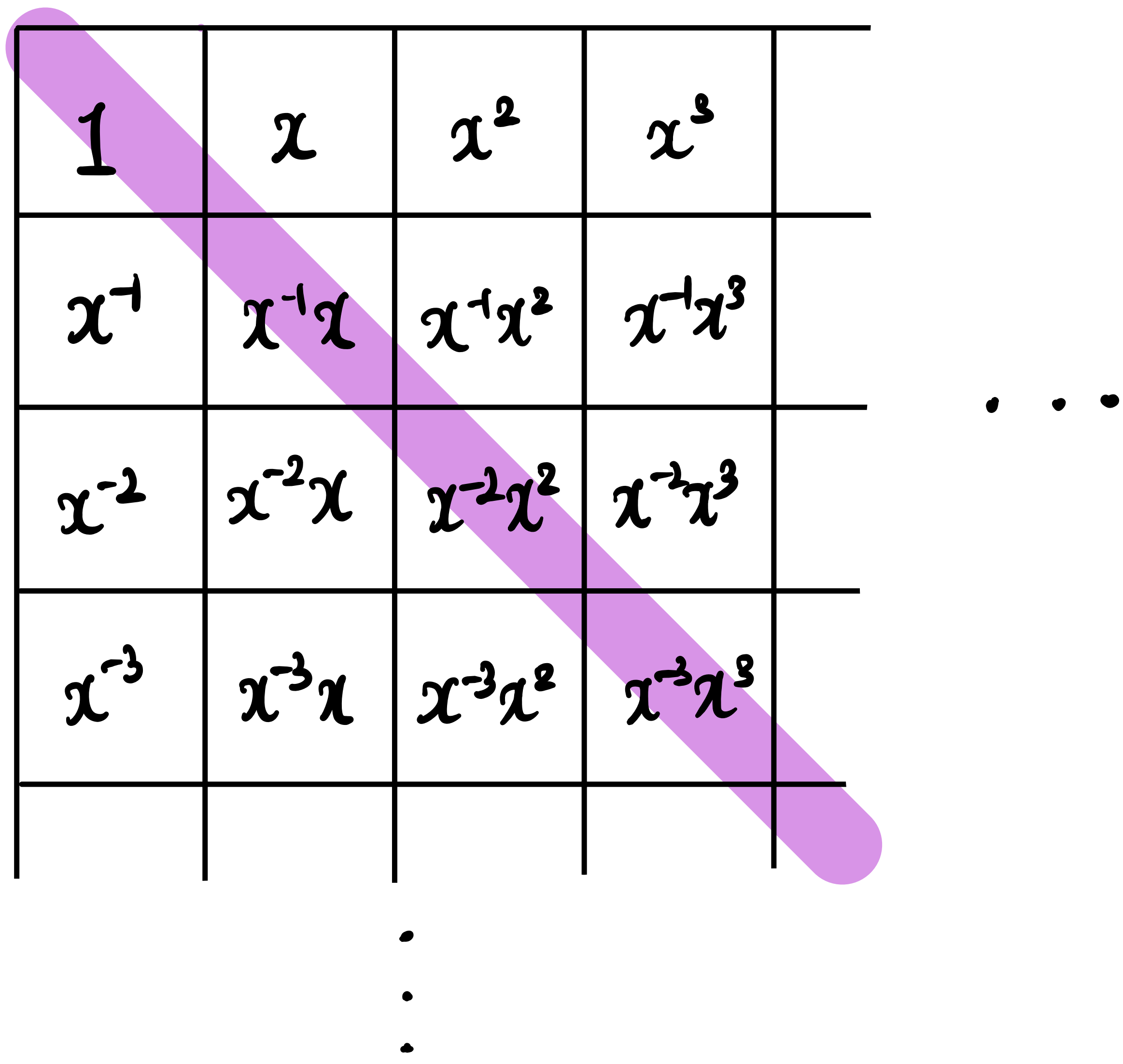
A subsemigroup of **B** is diagonal if it consists of diagonal elements only:



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A subsemigroup of **B** is upper (similarly lower) if it consists of upper diagonal or diagonal elements only:

1	x	x^2	x^3	
x^{-1}	$x^{-1}x$	$x^{-1}x^2$	$x^{-1}x^3$	
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A subsemigroup of **B** is square if it contains elements both above and below diagonal:

1	x	x^2	x^3	
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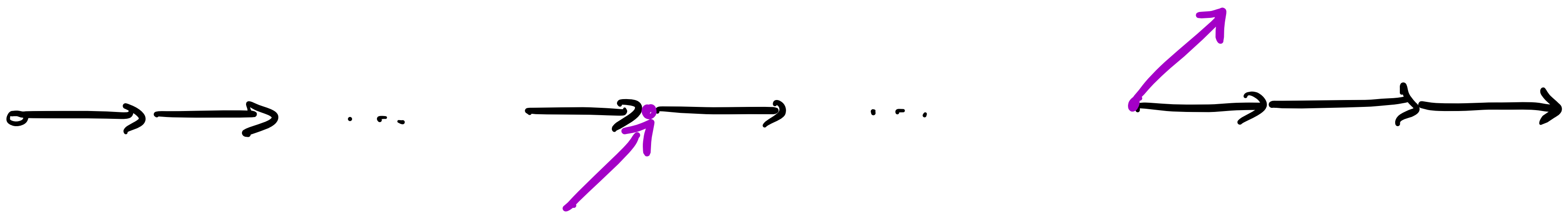

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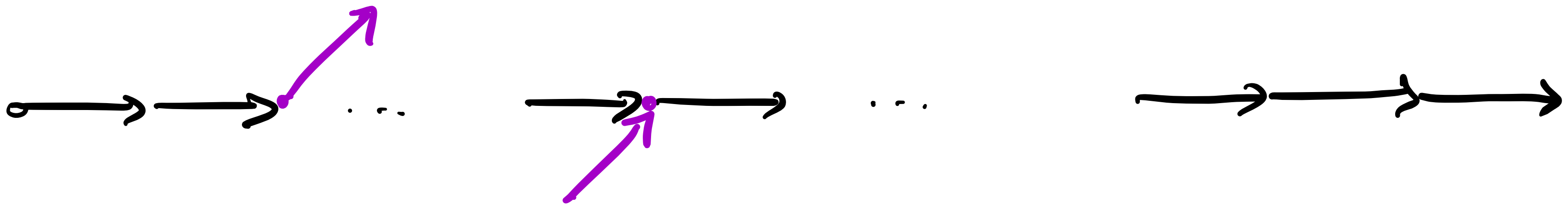

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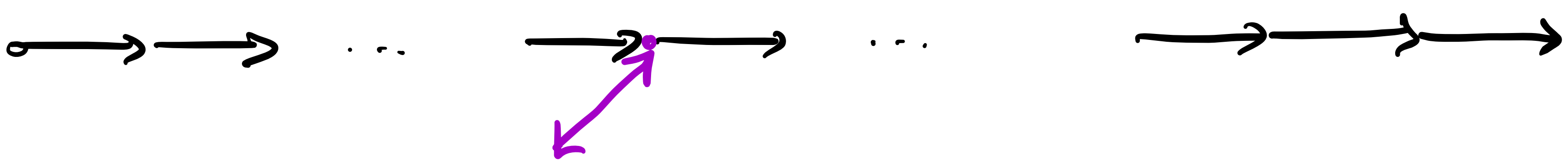

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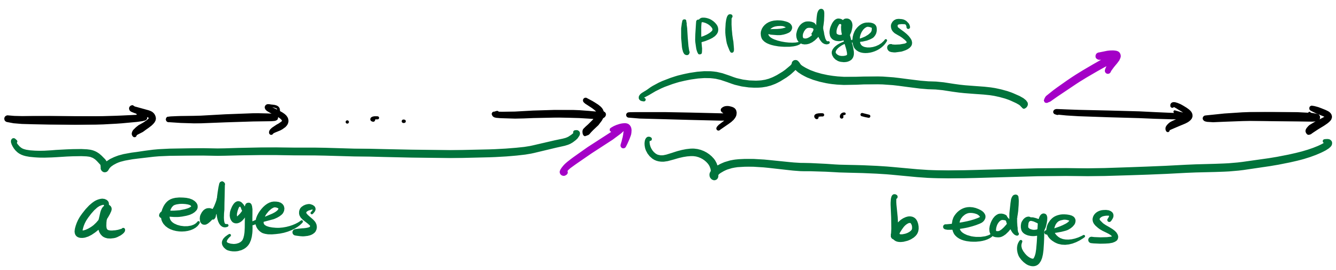

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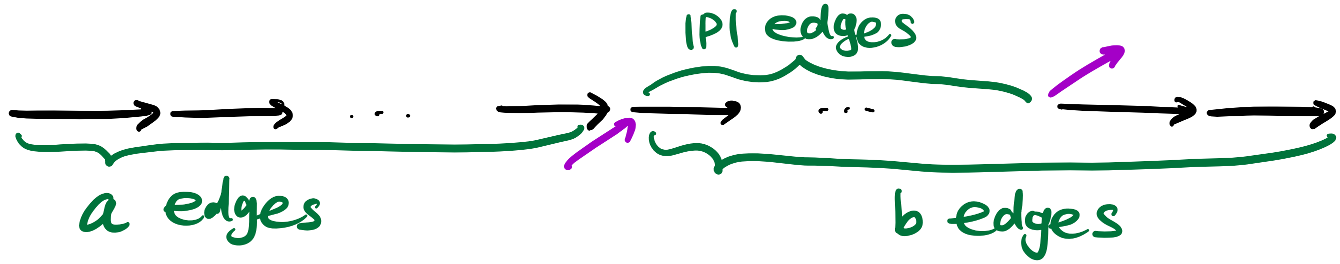

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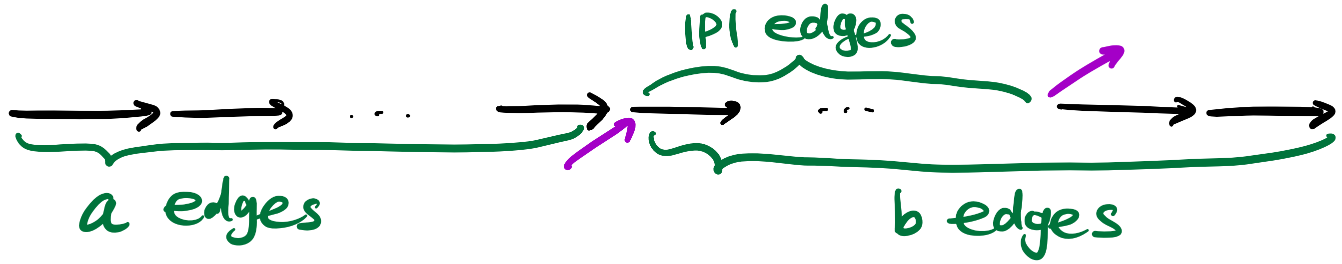

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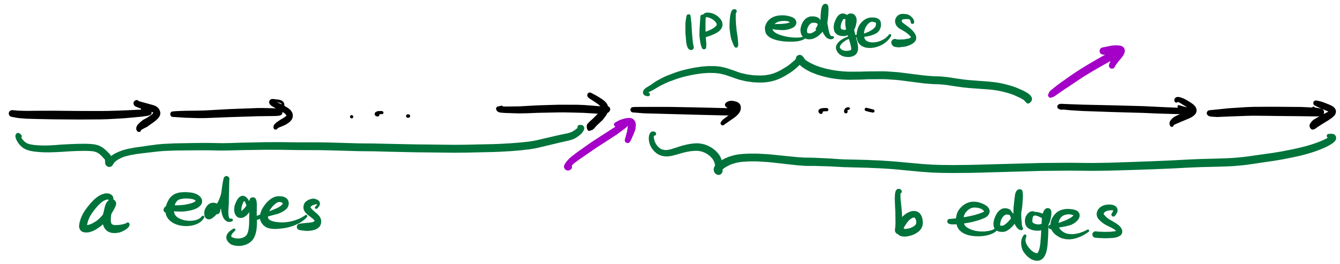

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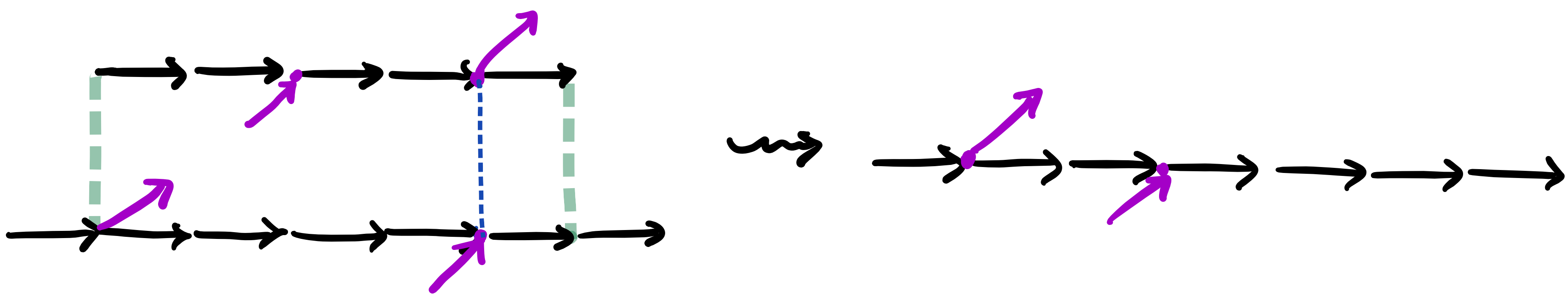

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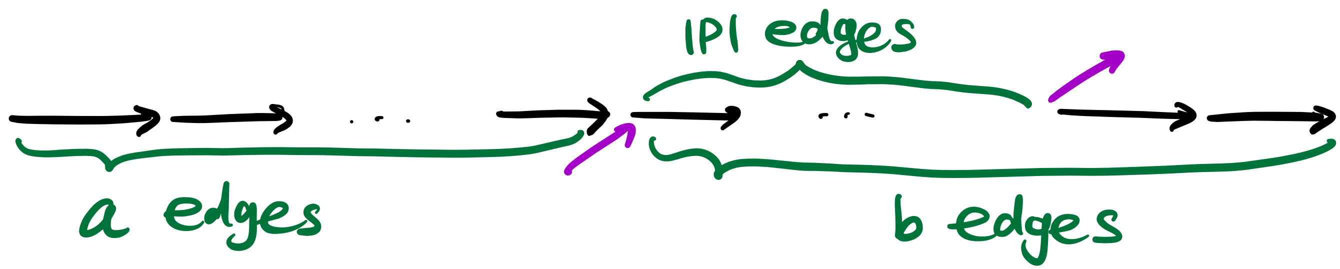

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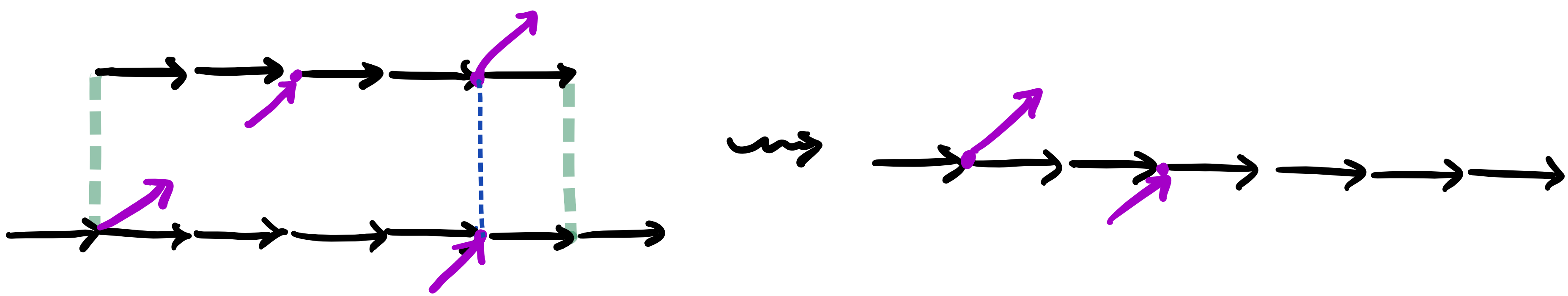

$$(-a_1,\ p_1,\ b_1)(-a_2,\ p_2,\ b_2) =$$
$$(-\max(a_1, a_2 - p_1),\ p_1 + p_2,\ \max(b_1, b_2 + p_1))$$

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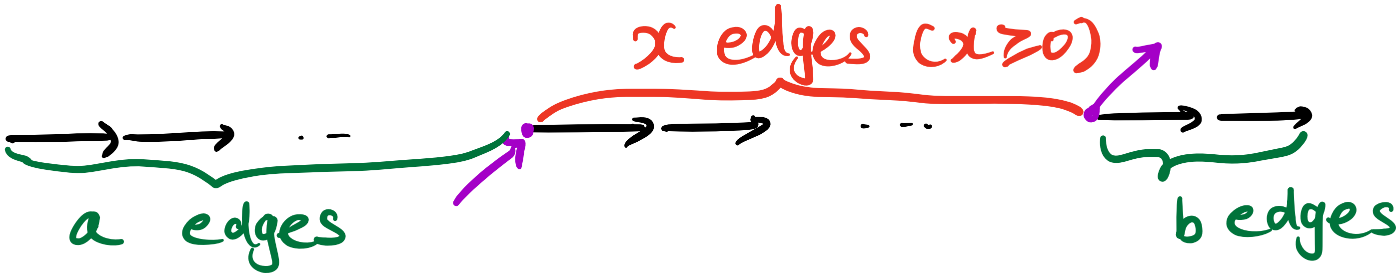

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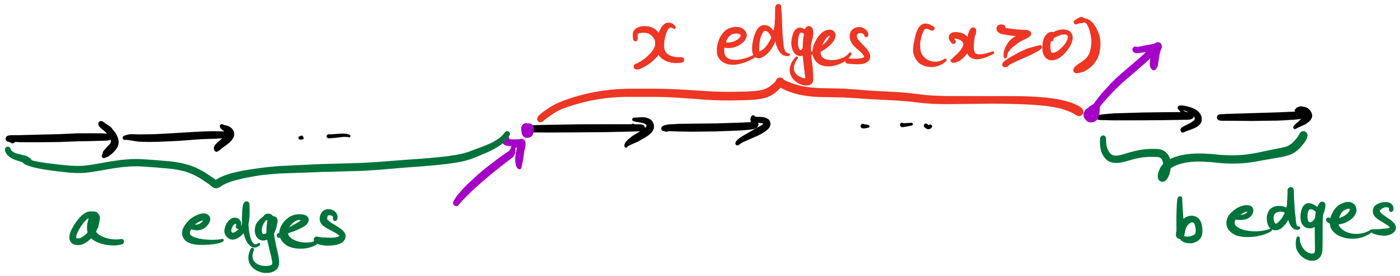

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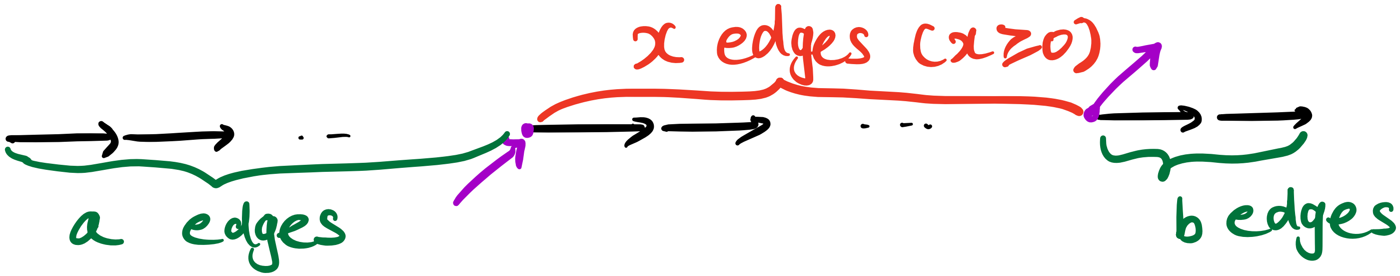

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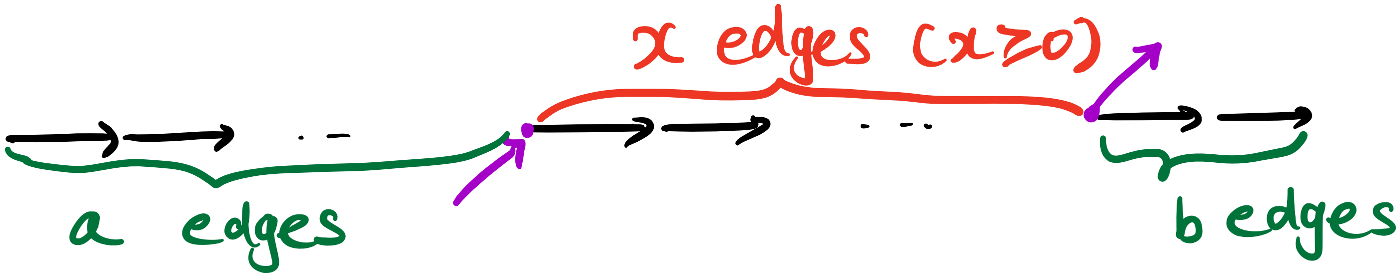

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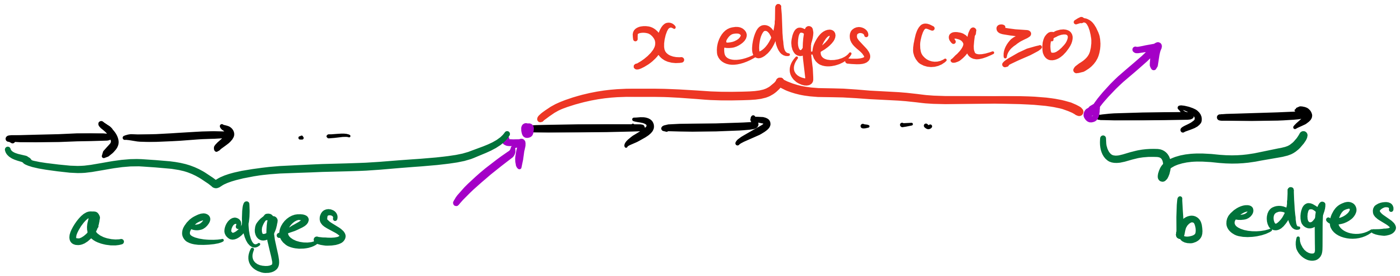

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$\rightsquigarrow$ if $S, T \leq \mathrm{FI}$ are finitely generated subsemigroups and at least one is one-sided, then $S \cap T$ is finitely generated.

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  Idea: find two finitely generated subsemigroups whose intersection intersect with infinitely many $\mathrm{PE}_{a,b}$.

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$\Longrightarrow$ FI does not have the semigroup Howson property.

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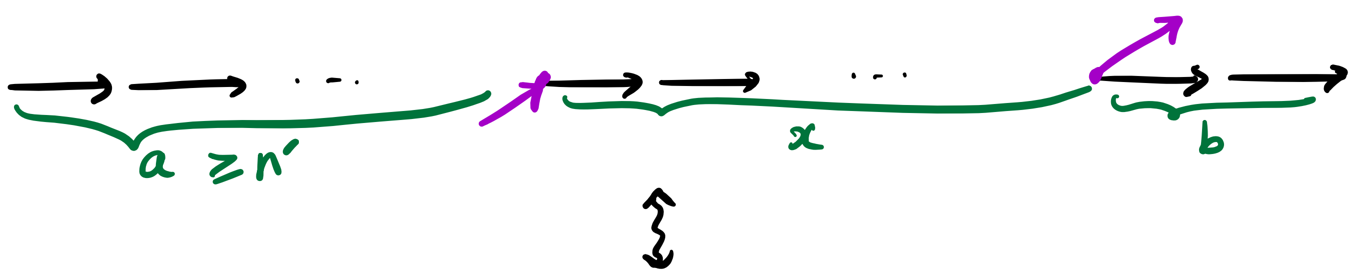

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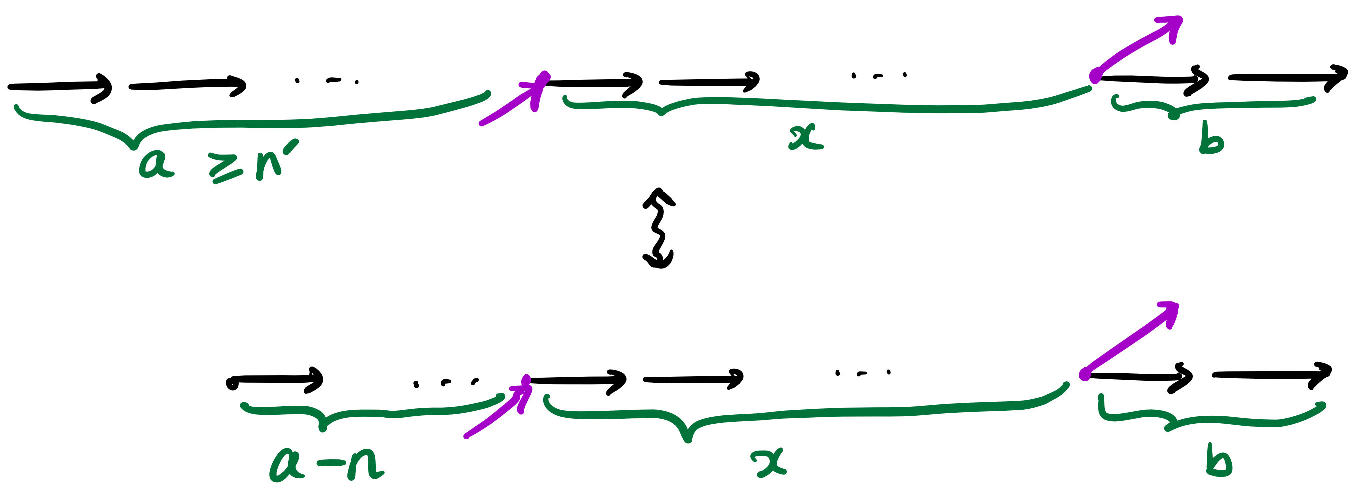

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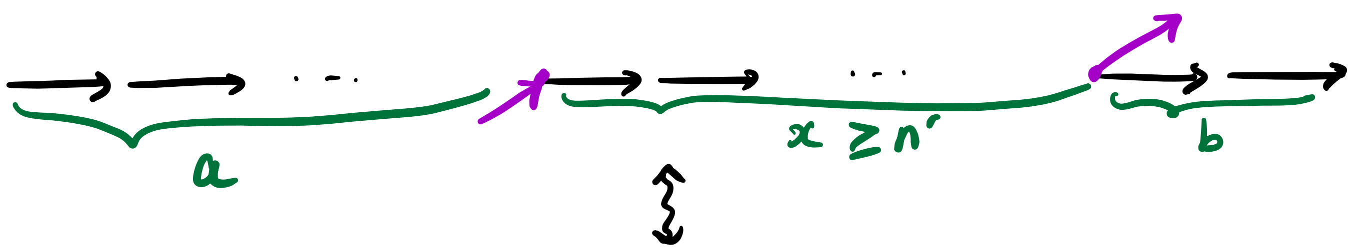

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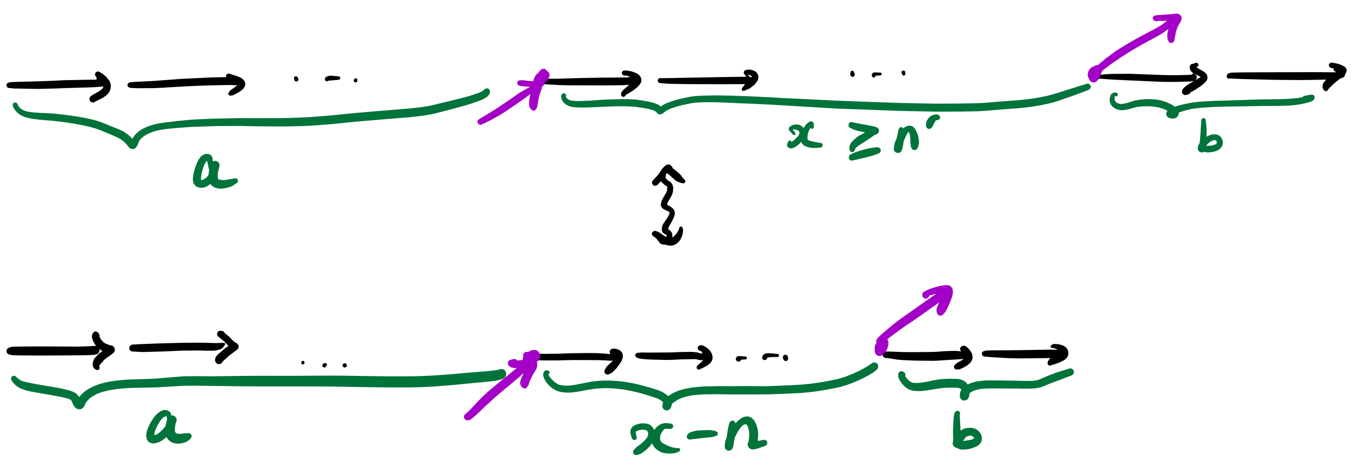

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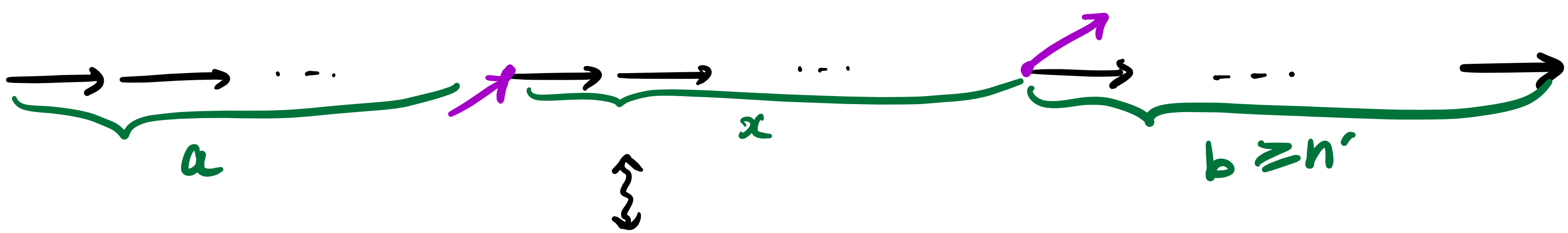

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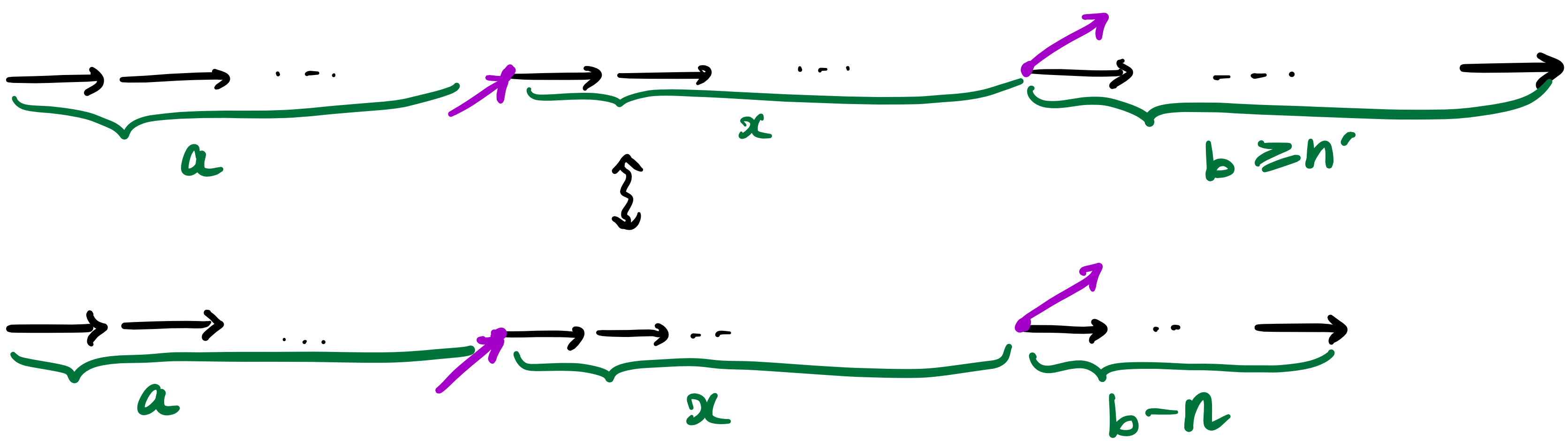

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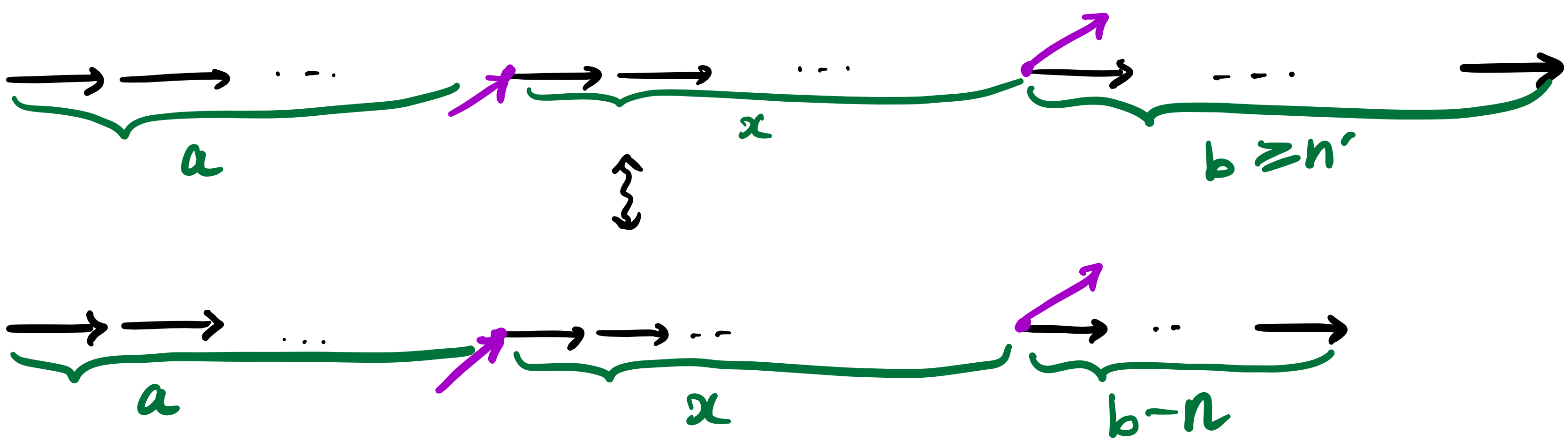

Note: Silva proved above for rational subsets.

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- Finitely generated two-sided subsemigroup of FI are precisely those that have this 'periodic' behaviour in PE and in $\mathsf{NE} := \mathsf{N} \cup \mathsf{E}$ plus two other 'technical conditions'.
- An example of $S, T \leq \mathsf{FI}$ finitely generated, two-sided and whose intersection $S \cap T$ is non-finitely generated two-sided can be given by showing that the intersection $S \cap T$ fails one of the technical conditions.

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Thank you for listening!